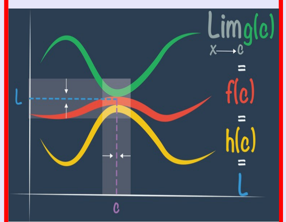
Math 261

Spring 2022

Lecture 26



Class QZ 15

Evaluate $\int_0^a x\sqrt{a^2-x^2}\,dx = \int_{a^2}^{0} \sqrt{u}\cdot\frac{du}{-2}$ ✓

$\boxed{u = a^2 - x^2}$

$du = -2x\,dx$

$\frac{du}{-2} = \boxed{x\,dx}$

$x=0 \rightarrow u=a^2$

$x=a \rightarrow u=0$

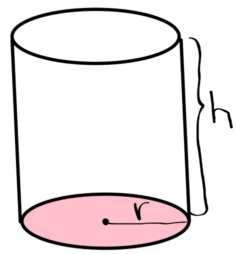
$= -\frac{1}{2}\cdot -\int_0^{a^2} u^{1/2}\,du$

$= \frac{1}{2}\cdot \frac{u^{3/2}}{3/2}\Big|_0^{a^2}$

$= \boxed{\frac{1}{3}a^3}$ ✓✓

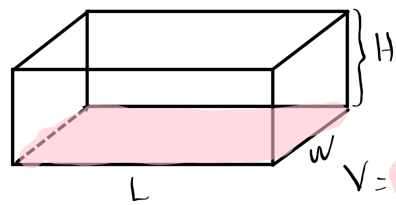
May 16, 2022

Intro. to Volume



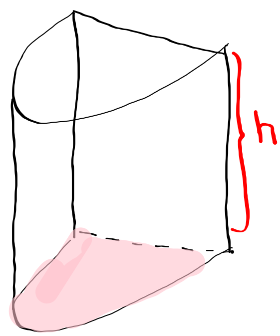
$V = \pi r^2 h$

Area of the base



$V = LWH$

Area of the base



Volume = Area of the base · Height

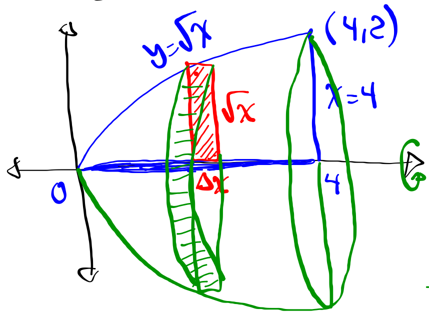
$$V = \lim_{n \to \infty} \sum_{i=1}^{n} A(x_i)\,\Delta x = \int_a^b A(x)\,dx$$

$$V = \lim_{n \to \infty} \sum_{i=1}^{n} A(y_i)\,\Delta y = \int_c^d A(y)\,dy$$

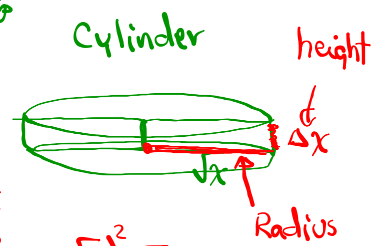
$A(x)$ or $A(y)$ are areas of cross-section base perpendicular to x-axis or y-axis.

Ex: Consider the region enclosed by

$y = \sqrt{x}$, $y = 0$, and $x = 4$.



Let's rotate this region about the x-axis.



base Circular with $r = \sqrt{x}$

Area of the base $\pi r^2 = \pi(\sqrt{x})^2 = \pi x$

$$V = \int_0^4 \pi x \, dx = \pi \int_0^4 x \, dx = \pi \cdot \frac{x^2}{2}\Big|_0^4 = \pi \cdot \frac{4^2}{2} - 0$$

$$= \boxed{8\pi}$$

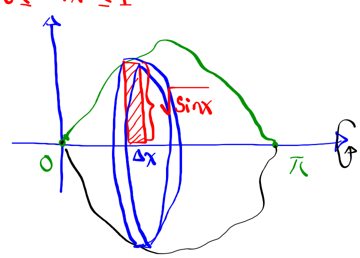
Consider the region bounded by $y = \sqrt{\sin x}$ and x-axis on $[0, \pi]$.

For $0 \leq x \leq \pi \Rightarrow 0 \leq \sin x \leq 1$

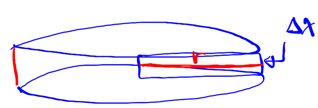
$x = 0$ $y = 0$

$x = \frac{\pi}{2}$ $y = 1$

$x = \pi$ $y = 0$



Let's rotate by x-axis.



$$V = \int_0^\pi \pi (\sqrt{\sin x})^2 dx$$

$$= \pi \int_0^\pi \sin x \, dx$$

$$= \pi \cdot (-\cos x)\Big|_0^\pi$$

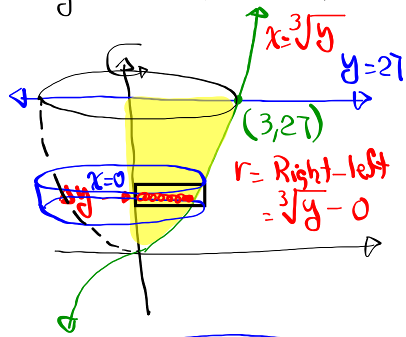
$A(x_i) \cdot \Delta x$

$\pi r^2 \Delta x$

$\pi (\sqrt{\sin x})^2 \Delta x$

$$= -\pi \left[\cos \pi - \cos 0\right] = -\pi[-1 - 1] = \boxed{2\pi}$$

Find the volume generated by rotating the region enclosed by $x=0$, $y=x^3$, and $y=27$ by the Y-axis.



r= Right-left
$=\sqrt[3]{y}-0$

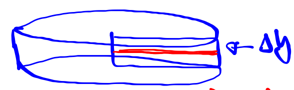
$$V=\int_0^{27}\pi(\sqrt[3]{y})^2\,dy$$

$$=\pi\int_0^{27} y^{2/3}\,dy$$

$$=\pi\cdot\frac{y^{5/3}}{5/3}\Bigg|_0^{27}$$

$$=\frac{3\pi}{5}\left[27^{5/3}-0\right]$$

$$=\frac{3\pi}{5}\cdot 3^5=\frac{\pi\, 3^6}{5}=\boxed{\frac{729\pi}{5}}$$



Volume $=A(y)\cdot\Delta y$

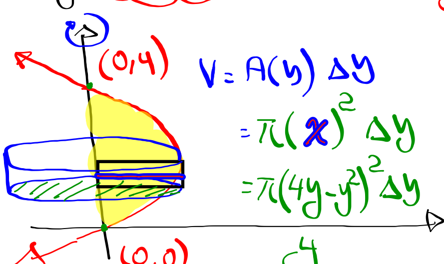
$=\pi(\sqrt[3]{y})^2\cdot\Delta y$

Find the volume generated by rotating the region enclosed by $x=4y-y^2$, and $x=0$ by y-axis. A.O.R.

Sideway Parabola opens left

X-Int (0,0)

Y-Int (0,0), (0,4)



$V=A(y)\,\Delta y$

$=\pi(x)^2\,\Delta y$

$=\pi(4y-y^2)^2\,\Delta y$

$$V=\int_0^4\pi(4y-y^2)^2\,dy$$

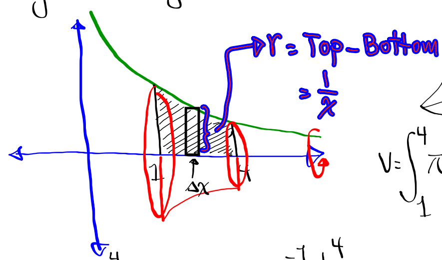
$$=\pi\int_0^4(16y^2-8y^3+y^4)\,dy$$

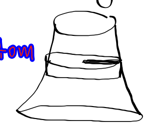
$$=\pi\left[\frac{16y^3}{3}-\frac{8y^4}{4}+\frac{y^5}{5}\right]\Bigg|_0^4$$

$$=\pi\left[\frac{16\cdot4^3}{3}-2(4)^4+\frac{4^5}{5}\right]=\boxed{\frac{512\pi}{15}}$$

Consider the region enclosed by $y=0$, $y=\frac{1}{x}$, $x=1$, and $x=4$. Find the volume generated by rotating the above region by x-axis

A.O.R.





$$V=\int_1^4 \pi\left(\frac{1}{x}\right)^2 dx$$

$$V=\pi\int_1^4 x^{-2}dx=\pi\cdot\frac{x^{-1}}{-1}\Big|_1^4=-\pi\cdot\frac{1}{x}\Big|_1^4$$

$$=-\pi\left[\frac{1}{4}-1\right]=-\pi\cdot\frac{-3}{4}$$

$$=\boxed{\frac{3\pi}{4}}$$

Name of the method:

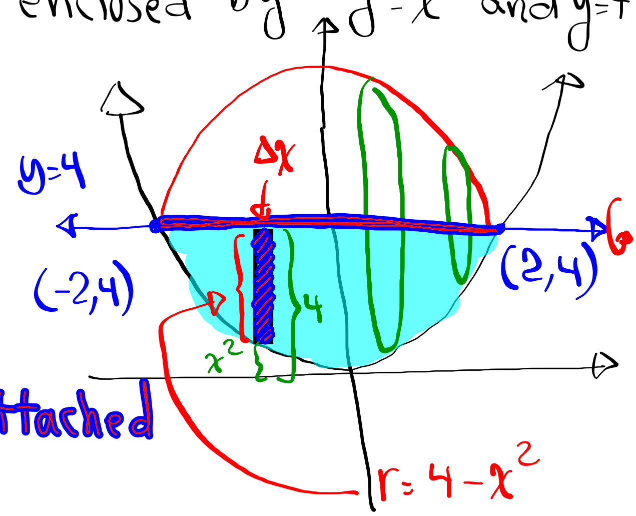
Disk

when to use disk method

when enclosed region most be totally attached to the A.O.R. and cross-section must be Perpendicular To A.O.R.

Rotate the region enclosed by $y=x^2$ and $y=4$ about $y=4$.

A.O.R.



Region totally attached to A.O.R.

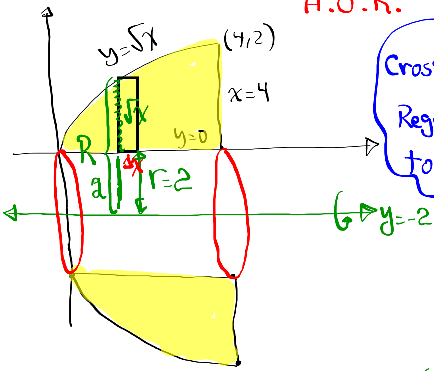
Cross-Section ⊥ A.O.R.

$$V=\int_{-2}^{2}\pi(4-x^2)^2dx$$

Answer $\boxed{\frac{512\pi}{15}}$

Consider the region enclosed by $y=\sqrt{x}$, $x=4$, and $y=0$. Find the volume by rotating above region by $y=-2$.

A.O.R.



Cross-Section ⊥ A.O.R.
Region is not totally attached to A.O.R.

Washer Method

$R^2 - r^2$

$(\sqrt{x}+2)^2 - 2^2$

$$V=\int_0^4 \pi\left[(\sqrt{x}+2)^2-2^2\right]dx$$

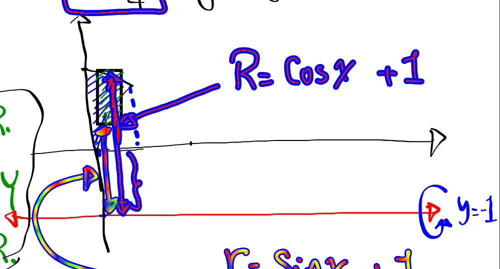
$$=\pi\int_0^4 (x+4\sqrt{x})\,dx=\pi\left[\frac{x^2}{2}+\frac{4x^{3/2}}{3/2}\right]_0^4$$

$$=\pi\left[8+\frac{2}{3}\cdot 4\cdot 4^{3/2}\right]=\pi\left[8+\frac{64}{3}\right]=\boxed{\frac{88\pi}{3}}$$

Rotate the region enclosed by $\boxed{y=\sin x}$, $\boxed{y=\cos x}$, $\boxed{x=0}$ and $\boxed{x=\frac{\pi}{4}}$ by $y=-1$.

Find the volume.

Cross-Section ⊥ A.O.R.
Region is not totally attached to A.O.R.

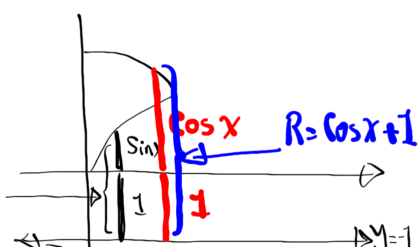


Washer Method

$R^2 - r^2$

$$V=\int_0^{\frac{\pi}{4}} \pi\left[R^2-r^2\right]dx$$

$r=\sin x+1$



$$V=\pi\int_0^{\pi/4}\left[(1+\cos x)^2-(1+\sin x)^2\right]dx$$

Finish by Wednesday

Consider the region bounded by $x=y^2$ and $x=1-y^2$.

Rotate this region by $x=3$, then find its Volume.

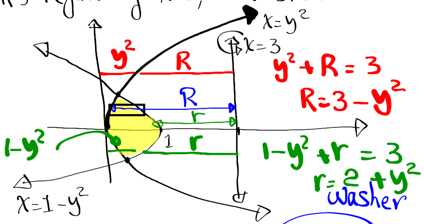
$y^2=1-y^2$

$2y^2=1$

$y^2=\frac{1}{2}$

$y=\pm\sqrt{\frac{1}{2}}$

$y=\pm\frac{\sqrt{2}}{2}$



$y^2+R=3$

$R=3-y^2$

$1-y^2+r=3$

$r=2+y^2$

Washer

Is cross-Section $\perp$ to A.O.R.? Yes

Is the region totally attached to A.O.R? NO

R^2-r^2

$R=3-y^2$

$r=2+y^2$

$$V=\int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}}\pi\left[R^2-r^2\right]dy=\pi\int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}}\underbrace{\left[(3-y^2)^2-(2+y^2)^2\right]}_{\text{even}}dy$$

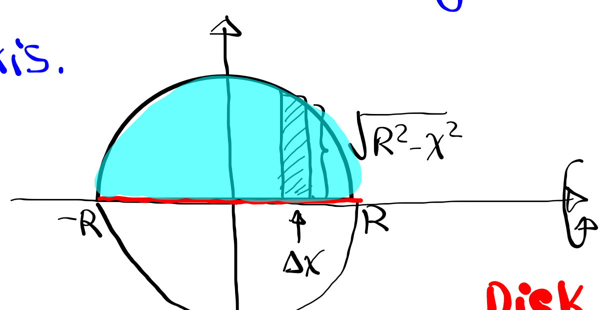
$$=\pi\cdot 2\int_0^{\frac{\sqrt{2}}{2}}\left[9-6y^2+\cancel{y^4}-4-4y^2-\cancel{y^4}\right]dy$$

$$=2\pi\int_0^{\frac{\sqrt{2}}{2}}(5-10y^2)dy$$

Finish by Wed.

Rotate the region bounded by $f(x)=\sqrt{R^2-x^2}$ and x-axis, by rotating it by x-axis.

Top half of the circle centered at $(0,0)$ with radius R.



Disk

πr^2

$$V=\int_{-R}^{R}\pi\left[\sqrt{R^2-x^2}\right]^2dx$$

Make sure to finish by Wed.

Class QZ 16

Find the average value of $f(x) = \cos^4 x \sin x$ on $[0, \pi]$. {Exact answer only.}

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{\pi - 0}\int_0^{\pi} \cos^4 x \sin x\, dx$$

$u = \cos x$
$du = -\sin x\, dx$

$x = 0, u = 1$
$x = \pi, u = -1$

$$= \frac{1}{\pi}\int_1^{-1} u^4 \cdot (-du) = \frac{1}{\pi}\int_{-1}^{1} u^4\, du$$

$$= \frac{2}{\pi}\int_0^1 u^4\, du = \frac{2}{\pi}\cdot\frac{u^5}{5}\Big|_0^1 = \boxed{\frac{2}{5\pi}}$$